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Small eigenvalues of large Hankel matrices

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Abstract. In this paper we investigate the smallest eigenvalue, denoted as λ_N, of a $(N+1)\times(N+1)$ Hankel or moments matrix, associated with the weight, $w(x) = \exp(-x^\beta)$, $x > 0$, $\beta > 0$, in the large N limit. Using a previous result, the asymptotics for the polynomials, $P_n(z)$, $z \notin [0, \infty)$, orthonormal with respect to w, which are required in the determination of λ_N are found. Adopting an argument of Szegö the asymptotic behaviour of λ_N, for $\beta > \frac{1}{2}$ where the related moment problem is determinate, is derived. This generalizes the result given by Szegö for $\beta = 1$. It is shown that for $\beta > \frac{1}{2}$ the smallest eigenvalue of the infinite Hankel matrix is zero, while for $0 < \beta < \frac{1}{2}$ it is greater then a positive constant. This shows a phase transition in the corresponding Hermitian random matrix model as the parameter β varies with $\beta = \frac{1}{2}$ identified as the critical point. The smallest eigenvalue at this point is conjectured.

1. Introduction

In the theory of Hermitian random matrices, the Hankel determinant plays an important role,

$$D_N = \det_{0\leqslant i,j\leqslant N}(\mu_{i+j}). \tag{1.1}$$

For a given weight function $w(t)$ on J ($\subseteq \mathbb{R}$), the moments μ_k are

$$\mu_k := \int_J w(t)t^k \,\mathrm{d}t \qquad k = 0, 1, 2, \ldots.$$

Associated with $w(t)$ is a Hankel matrix or moment matrix of order $N+1$, $\{H_{jk}\}$, whose entries are given by

$$H_{jk} := \mu_{j+k} \qquad 0 \leqslant j \quad k \leqslant N. \tag{1.2}$$

It is believed that correlations between eigenvalues of random matrices are universal after a suitable rescaling. In the following treatment we will show that a fundamental quantity, namely the least eigenvalues of these Hankel matrices exhibit a critical dependence on the weight function. It is this non-universal property that motivates our investigation of this problem.

If J is a single interval say $[a, b]$, where a and b are fixed and the Szegö condition,

$$\int_a^b \frac{v(x)\,\mathrm{d}x}{\sqrt{(b-x)(x-a)}} < \infty \qquad v := -\ln w$$

is satisfied, then the asymptotic behaviour of the Hankel determinants for large N is established, as shown by Szegö [10]. Let λ_N denote the smallest eigenvalue. Szegö also investigated the behaviour of λ_N for large N [8]. He studied the cases for which J can either be a finite or infinite

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interval with special choices for w. If $w(x) = 1, x \in (-1, 1)$ and $w(x) = 1, x \in (0, 1)$, then the respective smallest eigenvalues are for large N†

$$\lambda_N \simeq 2^{\frac{9}{4}} \pi^{\frac{3}{2}} N^{\frac{1}{2}} (\sqrt{2} - 1)^{2N+3}$$
$$\lambda_N \simeq 2^{\frac{15}{4}} \pi^{\frac{3}{2}} N^{\frac{1}{2}} (\sqrt{2} - 1)^{4N+4}.$$

Widom and Wilf [11] generalized Szegö's results to a kind of 'universal' law. Thus, if $w(x) > 0, x \in [a, b]$ and the Szegö condition is satisfied, then it was found in [11]

$$\lambda_N \simeq A N^{\frac{1}{2}} B^{-N}$$

where A and B are computable constants depending on w, a, b, and are independent of N.

In [8], Szegö also considered the cases of infinite intervals where $w(x) = \exp[-x^2], x \in (-\infty, +\infty)$ and $w(x) = \exp[-x], x \in [0, +\infty)$, are the weights of the Hermite and Laguerre polynomials‡. The respective smallest eigenvalues are

$$\lambda_N \simeq 2^{\frac{13}{4}} \pi^{\frac{3}{2}} \mathrm{e} N^{\frac{1}{4}} \exp[-2(2N)^{\frac{1}{2}}]$$
$$\lambda_N \simeq 2^{\frac{7}{2}} \pi^{\frac{3}{2}} \mathrm{e} N^{\frac{1}{4}} \exp[-4N^{\frac{1}{2}}].$$

Observe that in the examples given above the smallest eigenvalues are exponentially small. Therefore, it is very hard to numerically invert the Hankel matrices associated with these weights.

It is well known that λ_N is given by the Rayleigh quotient

$$\lambda_N = \min \left\{ \frac{\sum_{j,k=0}^{N} H_{jk} x_j \overline{x}_k}{\sum_{j=0}^{N} |x_j|^2} \right\}. \tag{1.3}$$

If $\pi_N(z)$ is a polynomial of degree N, with coefficients x_j, $j = 0, \ldots, N$

$$\pi_N(z) := \sum_{j=0}^{N} x_j z^j \tag{1.4}$$

then

$$\sum_{j,k=0}^{N} H_{jk} x_j \overline{x}_k = \int_J |\pi_N(t)|^2 w(t)\, \mathrm{d}t \tag{1.5}$$

and

$$\sum_{j=0}^{N} |x_j|^2 = \int_0^{2\pi} |\pi_N(\mathrm{e}^{\mathrm{i}\phi})|^2 \frac{\mathrm{d}\phi}{2\pi}. \tag{1.6}$$

Consequently, we can rephrase the extremal expression for λ_N, (1.3), as

$$\frac{2\pi}{\lambda_N} = \max \left\{ \int_0^{2\pi} |\pi_N(\mathrm{e}^{\mathrm{i}\phi})|^2 \, \mathrm{d}\phi : \int_J |\pi_N(t)|^2 w(t)\, \mathrm{d}t = 1 \right\}. \tag{1.7}$$

Letting $\{P_n(t)\}$ be the polynomials, orthonormal with respect to $w(t)$, then π_N has the expansion

$$\pi_N(z) = \sum_{j=0}^{N} c_j P_j(z). \tag{1.8}$$

Thus

$$\int_0^{2\pi} |\pi_N(\mathrm{e}^{\mathrm{i}\phi})|^2 \, \mathrm{d}\phi = \sum_{j,k=0}^{N} K_{jk} c_j \overline{c}_k \tag{1.9}$$

† Throughout this paper, the relation, $a_N \simeq b_N$ means $\lim_{N\to\infty} a_N/b_N = 1$.
‡ There is a factor of 4 missing from the original formula for λ_N; the last equation on p 677 of [8].

where

$$K_{jk} := \int_0^{2\pi} P_j(z)\overline{P_k(z)}\,\mathrm{d}\phi \qquad z = \mathrm{e}^{\mathrm{i}\phi}. \tag{1.10}$$

Therefore, (1.7) is equivalent to

$$\frac{2\pi}{\lambda_N} = \max\left\{\sum_{j,k=0}^{N} K_{jk}c_j\bar{c}_k : \sum_{j=0}^{N}|c_j|^2 = 1\right\}. \tag{1.11}$$

With the Schwarz inequality, which states that for all values of j and k

$$|K_{jk}| \leqslant K_{jj}^{\frac{1}{2}}K_{kk}^{\frac{1}{2}}$$

and Cauchy's inequality we obtain an upper bound of (1.11):

$$\begin{aligned}\sum_{j,k=0}^{N} K_{jk}c_j\bar{c}_k &\leqslant \sum_{j,k=0}^{N}|K_{jk}||c_j||c_k| \\ &\leqslant \sum_{j,k=0}^{N} K_{jj}^{\frac{1}{2}}K_{kk}^{\frac{1}{2}}|c_j||c_k| \\ &\leqslant \left(\sum_{j=0}^{N}K_{jj}\right)\left(\sum_{j=0}^{N}|c_j|^2\right) \\ &= \sum_{j=0}^{N}K_{jj}.\end{aligned} \tag{1.12}$$

Therefore, a lower bound for the smallest eigenvalue λ_N is given by

$$\frac{2\pi}{\sum_{j=0}^{N}K_{jj}} \leqslant \lambda_N. \tag{1.13}$$

This paper is organized as follows. In section 2, by adopting a previous result [5], we obtain the asymptotic formula for the polynomials orthonormal with respect to $w(t) := \exp[-t^{\beta}]$, $\beta > \frac{1}{2}$, which is then employed in sections 3 and 4 for the determination of the large N behaviour of λ_N. In these sections we show, following [8], by an appropriate choice of the vector $\{c_j\}$, that the lower bound given by (1.13) is in fact an asymptotic estimate for large N. By a simple application of the Laplace method, $\sum_{j=0}^{N}K_{jj}$ is estimated. Thus the asymptotic form of λ_N follows. In order to test the accuracy of the theory, these results are checked against numerical calculations for various β and N, which were obtained using the Jacobi rotation algorithm [12] to reduce the Hankel matrix to diagonal form. This is found in section 5.

2. The weight $w(t) = \exp[-t^{\beta}], t \in [0, \infty)$

In this case, the moments are

$$\mu_n = \frac{1}{\beta}\Gamma\left(\frac{n+1}{\beta}\right). \tag{2.1}$$

In order to find a lower bound for the smallest eigenvalue good knowledge is required of the associated orthonormal polynomials $\{P_N(z)\}$, for N large and $z \notin (0, \infty)$. In [5], by applying the linear statistics formula for matrix ensembles together with the Heine determinant representation, asymptotic forms for the polynomials with weight $w(t) = \exp[-v(t)]$, where

$v(t)$ is an arbitrary convex function supported on $[0, \infty)$, are derived. The zeros of these polynomials are supported on $(a, b) \subset \mathbb{R}$. Here $a = 0$, whilst $b(N)$ follows from the condition that ensures that $P_N(t)$ has N roots on (a, b), one finds that [5],

$$b(N;\beta) = CN^{\frac{1}{\beta}} \qquad \text{where} \quad C = C(\beta) := 4\left[\frac{\Gamma^2(\beta)}{\Gamma(2\beta)}\right]^{\frac{1}{\beta}} N^{\frac{1}{\beta}}. \tag{2.2}$$

The normalized polynomials as $N \to \infty$ are found, using [5], to be

$$P_N(t) \simeq \frac{(-1)^N}{\sqrt{2\pi b}} \frac{\exp[-f(t) + (2N+1)\ln(\sqrt{\zeta} + \sqrt{1+\zeta})]}{[\zeta(1+\zeta)]^{\frac{1}{4}}} \qquad \zeta := -\frac{t}{b} \qquad t \notin [0, b] \tag{2.3}$$

where f is given by

$$f(t) := \frac{\sqrt{t(t-b)}}{2\pi} \int_0^b \frac{\mathrm{d}y}{y-t} \frac{y^\beta}{\sqrt{y(b-y)}} \qquad t \notin [0, b]. \tag{2.4}$$

From the definition and basic properties of the hypergeometric functions [7],

$$\begin{aligned} f(t) &= -\frac{N}{\beta - \frac{1}{2}} \sqrt{\zeta(1+\zeta)}\, {}_2F_1\left(1, 1-\beta; \frac{3}{2} - \beta; -\zeta\right) - \frac{(-t)^\beta}{2} \sec \pi\beta \\ &= -\frac{N}{\beta}\sqrt{\frac{\zeta}{1+\zeta}}\, {}_2F_1\left(1, \frac{1}{2}; \beta+1; \frac{1}{1+\zeta}\right). \end{aligned} \tag{2.5}$$

At this point note the dichotomy of the problem, the nature of the hypergeometric function dictates that whilst the first representation is more convenient in the large b limit, where $|\zeta| \ll 1$, it cannot be used when $\beta = n + \frac{1}{2}$, $n = 1, 2, \ldots$, necessitating the use of the second result of (2.5) in such instances.

Using the fact that

$$\ln\left(\sqrt{\zeta} + \sqrt{1+\zeta}\right) = \sqrt{\zeta}\, {}_2F_1\left(\tfrac{1}{2}, \tfrac{1}{2}; \tfrac{3}{2}; -\zeta\right) \tag{2.6}$$

we find,

$$(2N+1)\ln\left(\sqrt{\zeta} + \sqrt{1+\zeta}\right) \simeq \frac{(-t)^\beta}{\sqrt{\pi} C^\beta} \sum_{r=0}^{E[\beta - \frac{1}{2}]} (-1)^r a_r \zeta^{r + \frac{1}{2} - \beta} \tag{2.7}$$

where $E[n]$ denotes the integer part of n and

$$a_r := \frac{\Gamma(r + \frac{1}{2})}{(r + \frac{1}{2})\Gamma(r+1)}. \tag{2.8}$$

So the asymptotic expression of the polynomials for $t \notin (0, \infty)$, is,

$$P_N(t) \simeq \frac{(-1)^N \zeta^{\frac{1}{4}}}{\sqrt{-2\pi t}} \exp\left(-f(t) + \frac{(-t)^\beta}{C^\beta \sqrt{\pi}} \sum_{r=0}^{E[\beta - \frac{1}{2}]} (-1)^r a_r \zeta^{r + \frac{1}{2} - \beta}\right). \tag{2.9}$$

To make further progress we now consider separately the two possible cases, as identified above, for $\beta > \frac{1}{2}$.

3. $\beta \neq n + \frac{1}{2}, n = 1, 2, 3 \ldots$

When $\beta \neq n + \frac{1}{2}$, we use the first form for $f(t)$ in equation (2.5). The series expansion for the function ${}_2F_1(1, 1-\beta; \frac{3}{2}-\beta; -\zeta)$, valid for $|\zeta| < 1$, is

$$ {}_2F_1\left(1, 1-\beta; \frac{3}{2}-\beta; -\zeta\right) = \frac{\Gamma(\frac{3}{2}-\beta)}{\Gamma(1-\beta)} \sum_{r=0}^{\infty} (-1)^r \frac{\Gamma(1-\beta+r)}{\Gamma(\frac{3}{2}-\beta+r)} \zeta^r \tag{3.1} $$

whilst for $|\zeta| < 1$, $\sqrt{1+\zeta}$ may be written as

$$ \sqrt{1+\zeta} = \frac{-1}{2\sqrt{\pi}} \sum_{r=0}^{\infty} (-1)^r \frac{\Gamma(r-\frac{1}{2})}{\Gamma(r+1)} \zeta^r. \tag{3.2} $$

With this noted, the expansion for $f(t)$ as $\zeta \to 0$ is

$$ \begin{aligned} f(t) \simeq & -\frac{1}{2\sqrt{\pi}} \frac{\Gamma(\frac{1}{2}-\beta)}{\Gamma(1-\beta)} \left(\frac{-t}{C}\right)^{\beta} \sum_{r=0}^{E[\beta-\frac{1}{2}]} (-1)^r b_r \zeta^{r+\frac{1}{2}-\beta} \\ & - \frac{(-t)^{\beta}}{2} \sec \pi\beta \end{aligned} \tag{3.3} $$

where

$$ b_r := \sum_{s=0}^{r} \frac{\Gamma(s-\frac{1}{2})\Gamma(1-\beta+r-s)}{\Gamma(s+1)\Gamma(\frac{3}{2}-\beta+r-s)}. \tag{3.4} $$

Recall that $\zeta = -tC^{-1}N^{-\frac{1}{\beta}}$, and by the use of equation (2.9) we have,

$$ P_N(t) \simeq \frac{(-1)^N}{\sqrt{2\pi}} (-tCN^{\frac{1}{\beta}})^{-\frac{1}{4}} \exp\left[\frac{(-t)^{\beta}}{2} \sec \pi\beta\right] \exp\left[\frac{N^{1-\frac{1}{2\beta}}}{\sqrt{\pi C}} \sum_{r=0}^{E[\beta-\frac{1}{2}]} (-1)^r A_r \frac{(-t)^{r+\frac{1}{2}}}{(CN^{\frac{1}{\beta}})^r}\right] \tag{3.5} $$

with

$$ A_r := a_r + \frac{\Gamma(\frac{1}{2}-\beta)}{2\Gamma(1-\beta)} b_r. \tag{3.6} $$

Note that, with $\beta = 1$, we find $C = 4$ and $A_0 = 4\sqrt{\pi}$ and, consequently, recover the classical result for the Laguerre polynomials due to Perron [9],

$$ P_N(t) \simeq \frac{(-1)^N}{2\sqrt{\pi}} (-tN)^{-\frac{1}{4}} \exp\left[2\sqrt{-tN} + \frac{t}{2}\right] \qquad t \notin [0, \infty) \quad N \to \infty. \tag{3.7} $$

With $P_N(t)$ having the form (3.5), where $A_0 = \frac{4\sqrt{\pi}\beta}{2\beta-1}$ is positive for $\beta > \frac{1}{2}$, we observe that for sufficiently large j and k the dominant contributions to K_{jk} are from the arc of the unit circle around $t = -1$. Thus by fixing an arbitrary positive number ω and confining ourselves to values of j and k satisfying

$$ N - \omega N^{\frac{1}{2\beta}} \leqslant j, k \leqslant N \tag{3.8} $$

we have

$$ K_{jk} \simeq \int_{\pi-\varepsilon}^{\pi+\varepsilon} P_j(\mathrm{e}^{\mathrm{i}\phi}) P_k(\mathrm{e}^{-\mathrm{i}\phi})\, \mathrm{d}\phi. \tag{3.9} $$

Using the substitution $\theta = \phi - \pi$ and expanding the integrand for $|\theta| \ll 1$ gives the following:

$$K_{jk} \simeq \frac{(-1)^{j+k}}{2\pi\sqrt{C}} \mathrm{e}^{\sec\pi\beta} N^{-\frac{1}{2\beta}} \int_{-\varepsilon}^{\varepsilon} \exp\left[\frac{1}{\sqrt{\pi C}} \sum_{r=0}^{E[\beta-\frac{1}{2}]} (-1)^r \frac{A_r}{C^r} \right. \times\left[\left(1 - \frac{(2r+1)^2\theta^2}{8}\right)(j^{1-\frac{1}{2\beta}-\frac{r}{\beta}} + k^{1-\frac{1}{2\beta}-\frac{r}{\beta}}) \left. + \frac{(2r+1)\mathrm{i}\theta}{2}(j^{1-\frac{1}{2\beta}-\frac{r}{\beta}} - k^{1-\frac{1}{2\beta}-\frac{r}{\beta}})\right]\right] \mathrm{d}\theta. \tag{3.10}$$

Because $j^{1-\frac{1}{2\beta}+\frac{r}{\beta}} - k^{1-\frac{1}{2\beta}-\frac{r}{\beta}}$ remains bounded in the range specified by (3.8) we can disregard the linear term in θ in the integrand. This integral can then be approximated by extending the range of integration to the real axis, which does not affect the asymptotic behaviour, as contributions from $(-\infty, -\varepsilon)$ and (ε, ∞) are sub-dominant compared with those from $[-\varepsilon, \varepsilon]$ as $j, k \to \infty$. Therefore,

$$K_{jk} \simeq \frac{(-1)^{j+k}}{(\pi C)^{\frac{1}{4}}} A_0^{-\frac{1}{2}} \mathrm{e}^{\sec\pi\beta} N^{-\frac{1}{2}-\frac{1}{4\beta}} \exp\left[\frac{1}{\sqrt{\pi C}} \sum_{r=0}^{E[\beta-\frac{1}{2}]} (-1)^r \frac{A_r}{C^r} (j^{1-\frac{1}{2\beta}-\frac{r}{\beta}} + k^{1-\frac{1}{2\beta}-\frac{r}{\beta}})\right]. \tag{3.11}$$

From (3.11), we see that when j and k are sufficiently large and satisfy (3.8),

$$K_{jk} \simeq (-1)^{j+k} K_{jj}^{\frac{1}{2}} K_{kk}^{\frac{1}{2}}. \tag{3.12}$$

This is especially useful as it enables the determination of the large N behaviour of λ_N. By choosing the vector $\{c_j\}$, as in [8], such that

$$c_j = \begin{cases} \sigma \mathrm{e}^{\mathrm{i}\pi j} K_{jj}^{\frac{1}{2}} & \text{if} \quad E[N - \omega N^{\frac{1}{2\beta}}] \leqslant j \leqslant N \\ 0 & \text{if} \quad j < E[N - \omega N^{\frac{1}{2\beta}}] \end{cases} \tag{3.13}$$

where σ is a positive number determined by the condition

$$\sum_{j=0}^{N} |c_j|^2 = \sigma^2 \sum_{j=E[N-\omega N^{\frac{1}{2\beta}}]}^{N} K_{jj} = 1 \tag{3.14}$$

we find, using (3.12) and (3.14), that

$$\begin{aligned} \sum_{j,k=0}^{N} K_{jk} c_j \bar{c}_k &= \sum_{j,k=E[N-\omega N^{\frac{1}{2\beta}}]}^{N} \sigma^2 \mathrm{e}^{\mathrm{i}\pi(j-k)} K_{jk} K_{jj}^{\frac{1}{2}} K_{kk}^{\frac{1}{2}} \\ &\simeq \sigma^2 \left[\sum_{j=E[N-\omega N^{\frac{1}{2\beta}}]}^{N} K_{jj}\right]^2 \\ &= \sum_{j=E[N-\omega N^{\frac{1}{2\beta}}]}^{N} K_{jj}. \end{aligned} \tag{3.15}$$

Recalling equation (1.11), we see that since ω is arbitrarily large the asymptotic behaviour of the maximum, by virtue of the inequality (1.13), is well approximated by $\sum_{j=0}^{N} K_{jj}$. Therefore, we have shown that

$$\frac{2\pi}{\lambda_N} \simeq \sum_{j=0}^{N} K_{jj}. \tag{3.16}$$

The leading behaviour of this sum for large N is in turn found by replacing the sum by an integral and by applying the Laplace method, which in this context may be stated as follows.

If for $x \in [a, b]$, the real continuous function $\phi(x)$ has as its maximum the value $\phi(b)$, then as $N \to \infty$

$$\int_a^b f(x) \exp[N\phi(x)]\,\mathrm{d}x \simeq \frac{f(b)\exp[N\phi(b)]}{N\phi'(b)}. \tag{3.17}$$

A simple calculation gives the expression for λ_N,

$$\frac{2\pi}{\lambda_N} \simeq \frac{1}{4}\pi^{-\frac{1}{4}} C^{\frac{1}{4}} A_0^{-\frac{1}{2}} \mathrm{e}^{\sec\pi\beta} N^{-\frac{1}{2}+\frac{1}{4\beta}} \exp\left[\frac{2N^{1-\frac{1}{2\beta}}}{\sqrt{\pi C}} \sum_{r=0}^{E[\beta-\frac{1}{2}]} (-1)^r \frac{A_r}{C^r} N^{-\frac{r}{\beta}}\right]. \tag{3.18}$$

Putting $\beta = 1$, Szegö's classical result for the Laguerre weight is recovered:

$$\frac{2\pi}{\lambda_N} \simeq 2^{-\frac{5}{2}} \pi^{-\frac{1}{2}} \mathrm{e}^{-1} N^{-\frac{1}{4}} \exp[4\sqrt{N}]. \tag{3.19}$$

From (3.18) we see that the smallest eigenvalue is exponentially small for large N and is zero for the corresponding infinite Hankel matrix.

4. $\beta = n + \frac{1}{2}, n = 1, 2, \ldots$

In this section we investigate the case where $\beta = n + \frac{1}{2}$, $n \geqslant 1$. Such cases, as was explained previously, require the second form of $f(t)$ in (2.5). To obtain the asymptotic expansion for $f(t)$, we first note the following result for the hypergeometric function.

If $\beta = n + \frac{1}{2}$ with $n = 1, 2, \ldots$ then

$${}_2F_1\left(1, \frac{1}{2}; \beta + 1; x\right) = L_\beta \frac{(x-1)^{\beta-\frac{1}{2}}}{x^{\beta+\frac{1}{2}}} \left(\sqrt{x} \ln\left[\frac{1+\sqrt{x}}{1-\sqrt{x}}\right] + \sum_{r=1}^{\beta-\frac{1}{2}} \frac{1}{L_{r-\frac{1}{2}}} \left(\frac{x}{x-1}\right)^r\right) \tag{4.1}$$

where L_r is given by

$$L_r := \frac{r}{2\pi} C^r(r). \tag{4.2}$$

This is easily be proved by using an inductive argument, noting the following version of Gauss' recursion relations [7]

$$\begin{aligned} {}_2F_1\left(1, \frac{1}{2}; n + \frac{5}{2}; z\right) &= \frac{(n+\frac{3}{2})(z-1)}{(n+1)z} \left[{}_2F_1\left(1, \frac{1}{2}; n + \frac{3}{2}; z\right) - {}_2F_1\left(1, \frac{1}{2}; n + \frac{1}{2}; z\right)\right] \\ &\quad + \frac{n(n+\frac{3}{2})}{(n+1)(n+\frac{1}{2})}\, {}_2F_1\left(1, \frac{1}{2}; n + \frac{3}{2}; z\right) \end{aligned} \tag{4.3}$$

together with the fact that

$${}_2F_1\left(1, \frac{1}{2}; \frac{5}{2}; z\right) = \frac{3}{4} \frac{(z-1)}{z^{\frac{3}{2}}} \ln\left[\frac{1+\sqrt{z}}{1-\sqrt{z}}\right] + \frac{3}{2}z. \tag{4.4}$$

Therefore,

$$f(t) = \frac{(-1)^{\beta+\frac{1}{2}}}{2\pi} (-t)^\beta \left(\ln\left[\frac{\sqrt{1+\zeta}+1}{\sqrt{1+\zeta}-1}\right] + \sqrt{1+\zeta} \sum_{r=1}^{\beta-\frac{1}{2}} (-1)^r \frac{\zeta^{-r}}{L_{r-\frac{1}{2}}}\right). \tag{4.5}$$

Using (3.2), we find

$$f(t) \simeq \frac{(-1)^{\beta+\frac{1}{2}}(-t)^\beta}{2\pi} \ln\left[\frac{4}{\zeta}\right] + \frac{(-t)^\beta}{4\pi^{\frac{3}{2}}} \sum_{r=0}^{\beta-\frac{1}{2}} (-1)^r \delta_{\beta-\frac{1}{2}-r} \zeta^{r+\frac{1}{2}-\beta} \qquad |\zeta| \ll 1 \tag{4.6}$$

where

$$\delta_r := \sum_{s=1}^{\beta-\frac{1}{2}} \frac{\gamma_{s-r}}{L_{s-\frac{1}{2}}} \tag{4.7}$$

and

$$\gamma_r := \begin{cases} \dfrac{\Gamma(r-\frac{1}{2})}{\Gamma(r+1)} & \text{if} \quad r \geqslant 0 \\ 0 & \text{if} \quad r < 0. \end{cases} \tag{4.8}$$

Recalling $\zeta = -tC^{-1}N^{-\frac{1}{\beta}}$, the strong asymptotics of the polynomials for $t \notin [0, \infty)$ reads,

$$P_N(t) \simeq \frac{(-1)^N}{\sqrt{2\pi}}(-tCN^{\frac{1}{\beta}})^{\frac{1}{4}} \exp\left[\frac{(-1)^{\beta-\frac{1}{2}}(-t)^{\beta}}{2\pi} \ln\left(\frac{4CN^{\frac{1}{\beta}}}{-t}\right)\right] \\ \times \exp\left[\frac{N^{1-\frac{1}{2\beta}}}{\sqrt{\pi C}} \sum_{r=0}^{\beta-\frac{1}{2}} (-1)^r B_r \frac{(-t)^{r+\frac{1}{2}}}{(CN^{\frac{1}{\beta}})^r}\right] \tag{4.9}$$

where

$$B_r := a_r - \frac{L_\beta}{2\beta}\delta_{\beta-\frac{1}{2}-r}. \tag{4.10}$$

Note the appearance of the logarithm in the exponential. Since $B_0 = \frac{4\sqrt{\pi}\beta}{2\beta-1} > 0$ and using an argument similar to that in the previous section, we see that in determining K_{jk} the essential contribution comes from the arc in the vicinity of $t = -1$. As before, restricting j, k to the range given in (3.8), we have,

$$K_{jk} \simeq \int_{-\varepsilon}^{\varepsilon} P_j(-\mathrm{e}^{\mathrm{i}\theta})P_k(-\mathrm{e}^{-\mathrm{i}\theta})\,\mathrm{d}\theta. \tag{4.11}$$

We expand the exponential in the integrand for $|\theta| \ll 1$, keeping terms up to second order and then extend the range of integration to the infinite interval. Because $j^{1-\frac{1}{2\beta}-\frac{r}{\beta}} - k^{1-\frac{1}{2\beta}-\frac{r}{\beta}}$ and $\ln(j/k)$ remain bounded in the range given by (3.8), we find

$$K_{jk} \simeq \frac{(-1)^{j+k}}{(\pi C)^{\frac{1}{4}}} B_0^{-\frac{1}{2}} N^{-\frac{1}{2}-\frac{1}{4\beta}} (4CN^{\frac{1}{\beta}})^{\frac{(-1)^{\beta-\frac{1}{2}}}{\pi}} \\ \times \exp\left[\frac{1}{\sqrt{\pi C}} \sum_{r=0}^{\beta-\frac{1}{2}} (-1)^r \frac{B_r}{C^r}(j^{1-\frac{1}{2\beta}-\frac{r}{\beta}} + k^{1-\frac{1}{2\beta}-\frac{r}{\beta}})\right]. \tag{4.12}$$

Again, note that for sufficiently large j and k, satisfying (3.8),

$$K_{jj} \simeq (-1)^{j+k} K_{jj}^{\frac{1}{2}} K_{kk}^{\frac{1}{2}}. \tag{4.13}$$

Repeating the argument of the previous section, it follows that

$$\frac{2\pi}{\lambda_N} \simeq \int_0^N K_{jj}\,\mathrm{d}j. \tag{4.14}$$

The leading term in the asymptotic expansion of this integral as $N \to \infty$ follows from an application of the Laplace method and is given by

$$\frac{2\pi}{\lambda_N} \simeq \frac{1}{4}\pi^{-\frac{1}{4}} C^{\frac{1}{4}} B_0^{-\frac{1}{2}} N^{-\frac{1}{2}+\frac{1}{4\beta}} (4CN^{\frac{1}{\beta}})^{\frac{(-1)^{\beta-\frac{1}{2}}}{\pi}} \exp\left[\frac{2N^{1-\frac{1}{2\beta}}}{\sqrt{\pi C}} \sum_{r=0}^{\beta-\frac{1}{2}} (-1)^r \frac{B_r}{C^r} N^{-\frac{r}{\beta}}\right]. \tag{4.15}$$

Effectively, $\exp[\sec \pi\beta]$ in (3.18) is replaced by $(4CN^{1/\beta})^{\frac{(-1)^{\beta-1/2}}{\pi}}$. Note the alternating nature of this additional factor depending on whether $\beta - \frac{1}{2}$ is odd or even. Again, (4.15) shows that $\lim_{N\to\infty} \lambda_N = 0$. According to standard theory [1], the moment problem associated with $w(x)$, $x \geqslant 0$ is indeterminate if

$$\int_0^\infty \frac{v(x)}{\sqrt{x}(1+x)}\,\mathrm{d}x < \infty.$$

Therefore, $\beta = \frac{1}{2}$ is special as it marks the transition point at which the moment problem becomes indeterminate. Assuming, the result given in (2.9) holds, we have

$$P_N(t) \simeq \frac{(-1)^N}{2\pi}(-t)^{-\frac{1}{4}}N^{-\frac{1}{2}} \exp\left[\frac{\sqrt{-t}}{\pi}\left(\ln\left[\frac{4\pi N}{\sqrt{-t}}\right]+1\right)\right] \qquad t \notin [0,\infty). \tag{4.16}$$

Again, if we confine ourselves to the range where j and k are sufficiently large to enable the use of the above asymptotic representation, we find that the major contributions to K_{jk} are from the arc around $t = -1$. But, due to the behaviour of $P_N(t)$ with increasing N, it is quite clear that $|K_{jk}|$ decreases as $j, k \to \infty$, making an analysis analogous to that of the previous sections impossible.

It is, however, possible to obtain an approximate lower bound for the least eigenvalue, since (1.13) still holds. Applying the Christoffel–Darboux formula [9] and the result given in [4] for the large N off-diagonal recurrence coeeficients, we find,

$$\begin{aligned}\sum_{j=0}^N K_{jj} &= \int_{-\pi}^{\pi} \sum_{j=0}^N P_j(-\mathrm{e}^{\mathrm{i}\theta})P_j(-\mathrm{e}^{-\mathrm{i}\theta})\,\mathrm{d}\theta \\ &\simeq \pi^2 N^2 \int_{-\pi}^{\pi} \frac{P_N(-\mathrm{e}^{\mathrm{i}\theta})P_{N+1}(-\mathrm{e}^{-\mathrm{i}\theta}) - P_N(-\mathrm{e}^{-\mathrm{i}\theta})P_{N+1}(-\mathrm{e}^{\mathrm{i}\theta})}{\mathrm{e}^{\mathrm{i}\theta}-\mathrm{e}^{-\mathrm{i}\theta}}\,\mathrm{d}\theta.\end{aligned} \tag{4.17}$$

Thus, using the Laplace method,

$$\int_a^b \mathrm{d}x\; f(x)\exp[N\phi(x)] \simeq f(c)\exp[N\phi(c)]\sqrt{\frac{2\pi}{-N\phi''(c)}} \qquad \text{as} \quad N \to +\infty$$

where $c \in (a, b)$ is the maximum of $\phi(x)$ for $x \in (a, b)$, gives

$$\sum_{j=0}^N K_{jj} \simeq \frac{(4\pi Ne)^{2/\pi}}{4\sqrt{\ln(4\pi Ne)}}. \tag{4.18}$$

So at the point $\beta = \frac{1}{2}$ the smallest eigenvalue appears to decrease algebraically instead of exponentially.

5. Numerical results

In this section we check the accuracy of our asymptotic expressions for the least eigenvalue of the the various Hankel matrices against numerical results. Due to the fact that the moment matrices in these cases are very ill-conditioned because of the vast range in scale of the matrix elements, the Jacobi rotation algorithm [12], proved far more stable than the more conventional techniques for numerically determining a small selection of the eigenvalues of large symmetric matrices such as the Lanczos procedure or the Householder method [6]. This appears to be an unusual phenomenon. Because of the behaviour of the matrix elements in these problems it is necessary to implement a multiple-precision package that allows floating point arithmetic of arbitrary precision. The library of sub-routines created by Brent [3] was employed to combat the effect of rounding errors in the numerical procedures.

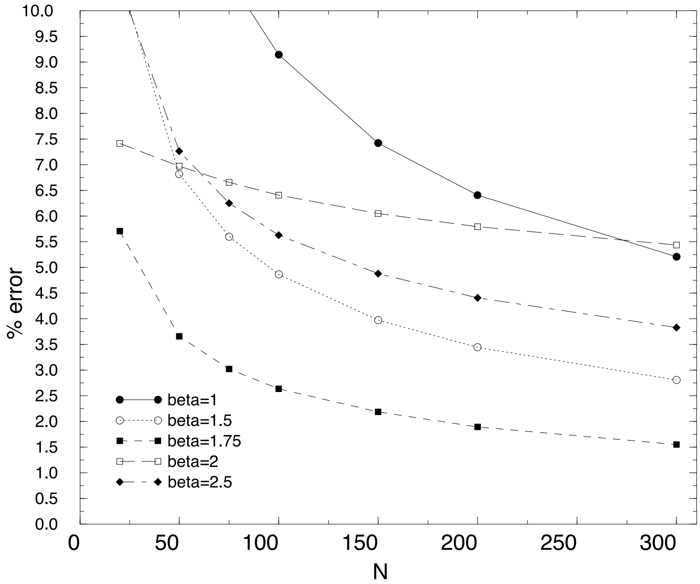

Figure 1. The percentage error of the theoretical values of λ_N when compared with those obtained numerically, for various β.

Table 1. Numerical and theoretical values of λ_N for various β.

β	N	Numerical λ_N	Theoretical λ_N
1	50	2.0948×10^{-10}	2.3695×10^{-10}
	100	2.1079×10^{-15}	2.3006×10^{-15}
	150	2.9551×10^{-19}	3.1743×10^{-19}
	200	1.6387×10^{-22}	1.7437×10^{-22}
	300	5.5215×10^{-28}	5.8090×10^{-28}
$\frac{3}{2}$	50	6.4066×10^{-22}	6.8438×10^{-22}
	100	6.2353×10^{-36}	6.5384×10^{-36}
	150	9.9476×10^{-48}	1.0343×10^{-47}
	200	2.8132×10^{-58}	2.9101×10^{-58}
	300	4.6009×10^{-77}	4.7300×10^{-77}
$\frac{7}{4}$	50	6.4483×10^{-27}	6.6844×10^{-27}
	100	1.6976×10^{-45}	1.7424×10^{-45}
	150	1.5193×10^{-61}	1.5525×10^{-61}
	200	3.9265×10^{-76}	4.0009×10^{-76}
	300	1.4844×10^{-102}	1.5074×10^{-102}
2	50	2.7356×10^{-31}	2.5449×10^{-31}
	100	3.8907×10^{-54}	3.6415×10^{-54}
	150	2.9557×10^{-74}	2.7769×10^{-74}
	200	8.9775×10^{-93}	8.4574×10^{-93}
	300	9.5593×10^{-127}	9.0396×10^{-127}
$\frac{5}{2}$	50	2.2384×10^{-38}	2.4010×10^{-38}
	100	1.2580×10^{-68}	1.3288×10^{-68}
	150	5.3195×10^{-96}	5.5789×10^{-96}
	200	1.2155×10^{-121}	1.2691×10^{-121}
	300	1.5236×10^{-169}	1.5819×10^{-169}

For $0 < \beta < \frac{1}{2}$, the corresponding moment problem becomes indeterminate [1], and as a consequence, the sum

$$\sum_{j=0}^{\infty} |P_j(z)|^2$$

converges for every z in every compact subset of the complex plane. Therefore,

$$\sum_{j=0}^{\infty} K_{jj} = \xi > 0$$

and the smallest eigenvalue for the corresponding infinite Hankel is a positive constant bounded below by $2\pi/\xi$. Proof of the extention of the above statement to all indeterminate moment problems and other related topics can be found in [2]. The situation for $0 < \beta < \frac{1}{2}$ is in contrast to the results for $\beta > \frac{1}{2}$ where (3.18) and (4.15), as confirmed by the numerics, show that the sum diverges—a fact that is also well known from the standard theory when the moment problem is determinate [1]. This separation of behaviour in the two regions is the phenomenon of phase transition alluded to earlier.

The comparison between the numerical values of λ_n and those obtained from the theoretical expressions (3.18) and (4.15) is shown in table 1 and figure 1.

References

[1] Akhiezer N I 1965 *The Classical Moment Problem and Some Related Questions in Analysis* (Edinburgh: Oliver and Boyd) (Engl. transl.)

[2] Berg C, Chen Y and Ismail M E H 1999 Small eigenvalues of large Hankel matrices: the indeterminate case *Preprint* xxx.soton.ac.uk/abs/math/9907110

[3] Brent R P 1981 *Technical Report* TR-CS-81-08, MP user's guide (4th edn) Australian National University

[4] Chen Y and Ismail M E H 1997 Thermodynamic relations the Hermitian matrix ensembles *J. Phys. A: Math. Gen.* **30** 6633–54

[5] Chen Y and Lawrence N 1998 On the linear statistics of Hermitian random matrices *J. Phys. A: Math. Gen.* **31** 1141–52

[6] Golub G H and Van Loan C F 1983 *Matrix Computations* (Baltimore, MD: Johns Hopkins University Press)

[7] Gradshteyn I S and Ryzhik I M 1994 *Table of integals, Series and Products* (New York: Academic)

[8] Szegö G 1982 On some Hermitian forms associated with two given curves of the complex plane *Collected Papers* vol 2 (Basle: Birkhäuser) p 666

[9] Szegö G 1975 *Orthogonal Polynomials* 4th edn (Providence, RI: American Mathematical Society)

[10] Szegö G 1982 Hankel forms (English translation of A Hankel—féle formákról) *Collected Papers* vol 1 (Basle: Birkhäuser) p 111

[11] Widom H and Wilf H S 1966 Small eigenvalues of large Hankel matrices *Proc. Am. Soc.* **17** 338–44

[12] William H *et al* 1989 *Numerical Recipes: The Art of Scientific Computing* (Cambridge: Cambridge University Press)